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# Constitutive Modelling of Static Liquefaction of Sand with Fines

M. M. Rahman[1], S. R. Lo[2] and Y. A. Dafalias[3]

## ABSTRACT

Two powerful propositions: state dependency of constitutive relation and equivalent granular void ratio, $e^*$ are combined to obtain a constitutive model that can capture the influence of fines content, $f_c$ (particle size<0.075mm) irrespective of density and stress states. This was achieved by merely substituting $e^*$ and equivalent granular state parameter, $\psi^*$ for void ratio, $e$ and state parameter, $\psi$ into a state dependent constitutive model for clean sands. While this model can capture static liquefaction behaviour, it is further evaluated in this article with counter-intuitive but significant features of the influence of $f_c$, irrespective of density and stress states.

## Introduction

Three types of undrained behaviour: non-flow (NF), limited flow (LF), and flow (F) are observed for sand under compressive loading as illustrated in Fig. 1a. The types of behaviour depend on initial states of sand i.e. void ratio ($e$), mean effective stress ($p'$); where $p' = (\sigma_1'+2\sigma_3')/3$, $\sigma_1'$ and $\sigma_3'$ are major and minor principal stresses, respectively. Flow (F) behaviour is observed for higher $e$ i.e. for loose sand where the deviator stress, $q = (\sigma_1'-\sigma_3')$, attains an 'initial peak' value, then strain softens to steady state (SS) (also known as Critical State (CS)) strength at constant $q$, $p'$ and pore water pressure ($u$). Limited flow (LF) behaviour is observed for medium dense sand where $q$ drops to a temporary SS known as quasi-steady state (QSS) (Ishihara, 1993) and then deviator stress continues with strain hardening until SS is reached. The point where effective stress path (ESP) turns to the right is called phase transformation (PT). Non-flow (NF) behaviour is observed for dense sand where strain hardening occurs throughout undrained shearing to SS. The SS data points from several tests form a line in $e$-log($p'$) space which is referred to as steady state line (SSL) hereafter. According to critical state soil mechanics (CSSM), the state of sand with respect to SSL provides an important framework for predicting sand's behaviour. A state index namely state parameter ($\psi$), defined by Been and Jefferies (1985) as the difference between void ratio at the current state and the SS at the same $p'$, provides such a relative measure of current state to SS as shown in Fig. 1b. The state dependency for sand's behaviour was captured in a constitutive model through $\psi$ first by Manzari and Dafalias (1997) and subsequently by Li and Dafalias (2000). However, CSSM framework for sand mixed with fines

[1] Visiting Scholar, Department of Civil and Environmental Engineering, University of California, Davis, CA, USA and Senior Lecturer, School of Natural and Built Environments, University of South Australia, Mawson Lakes, Adelaide, Australia, Mizanur.Rahman@unisa.edu.au

[2]Associate Professor, School of Engineering and Information Technology, University of New South Wales, ADFA Campus, Canberra, Australia, s.lo@adfa.edu

[3]Distinguished Professor, Department of Civil and Environmental Engineering, University of California, Davis, CA, USA and Emeritus Professor, National Technical University of Athens, 15773, Hellas, jfdafalias@ucdavis.edu

(particle size<0.075mm) is not as straight forward as clean sand. The SS data points for sand with different fines content, $f_c$ form different SSLs. The equivalent granular void ratio, $e^*$ forms a single trend of SS data points, irrespective of $f_c$, in $e^*$-log($p'$) space which lead to modification $\psi$ to equivalent state granular parameter, $\psi^*$ (Rahman, *et al.*, 2008). These two propositions: state dependency and equivalent granular state are recently combined in the above constitutive model by replacing $e^*$ and $\psi^*$ for $e$ and $\psi$, respectively, to capture the effect of fines (Rahman, *et al.*, 2014b). However, the model was not evaluated in detail with counter-intuitive but significant feature of undrained behaviour for sand with fines, such as- (i) $f_c$ can play a role as important as $e$ for sand with fines behaviour, (ii) same initial state of $\psi$ or $\psi^*$ may exhibits either F or LF behaviour (Rahman and Lo, 2014) and, (iii) LF associated with evolving states that crosses SSL intermediately and then turn back to SS. Therefore, the model is evaluated with these counter-intuitive but significant features of liquefaction behaviour irrespective of $f_c$.

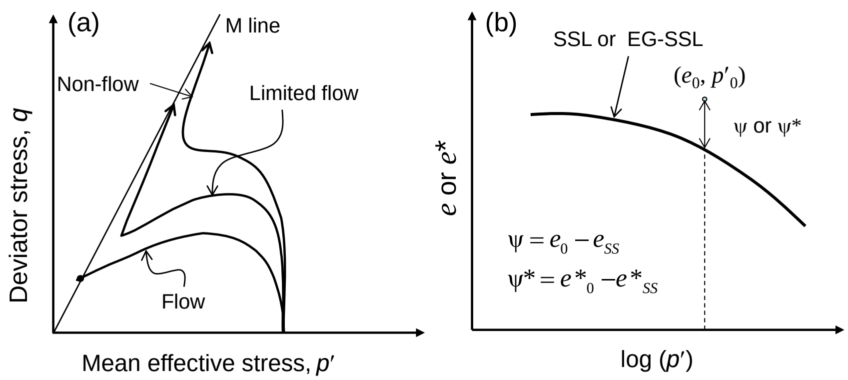


Figure 1: (a) undrained compression behaviour and, (b) definition of $\psi$ or $\psi^*$.

## Background of the Propositions

The characteristics of the aforementioned types of undrained behaviour of sands such as peak $q$, QSS and pore water pressure generation can also be correlated to $\psi$ (Jefferies and Been, 2006). Consequently, $\psi$ has been used in constitutive modeling of clean sands (Jefferies, 1993). Wood, *et al.* (1994) proposed that $\eta_f$, where subscript "f" denotes at peak-failure in effective stress-ratio, can be related to $\psi$. Along the same line of thinking, Manzari and Dafalias (1997) proposed that $\eta_{PT}$, where "PT" denotes at phase transformation, is also related to $\psi$; and this relationship directly affects the stress-dilatancy relationship. These two constitutive features were explicitly incorporated in the development of constitutive models for clean sand within the critical state framework so that the influence of both $e$ and $p'$ are captured in a unifying manner via $\psi$ (Manzari and Dafalias, 1997, Gajo and Muir Wood, 1999, Li and Dafalias, 2000, Dafalias and Manzari, 2004).

While the application of CSSM in constitutive modelling of sand behaviour is still evolving, the presence of $f_c$ in sands significantly alters it's behaviour and a consensus is yet to be reached. Some reported sand's shear resistance increase with increasing $f_c$, while other reported it decreases with increasing $f_c$. Note, different datasets used different density indexes, such as $e$, relative density ($D_R$) etc., as comparison basis and it is appeared that the impression of decrease or increase resistance was, partly, due to inconsistent comparison basis.

Inspired by the earlier work of Mitchell (1976), Thevanayagam, *et al.* (2002) defined equivalent void ratio, $e^*$ to account for the contribution of fines in sand's granular structure as following:

$$e^* = \frac{e + (1-b) f_c}{1-(1-b) f_c} \tag{1}$$

where $b$ represents the fraction of fines that are active in force transmission in the soil skeleton. This physical meaning of $b$ requires $1 \geq b \geq 0$. Earlier research suggested $b \approx 0$ when $f_c$ is low. At higher $f_c$, $b \neq 0$ and obtaining $b$ is a challenge. Rahman and co-workers, based on a re-interpretation of binary studies, proposed a semi-empirical equation for expressing $b$ as a function of $f_c$ and the size of fines relative to sand (Rahman, *et al.*, 2008, Rahman, *et al.*, 2009); and this enables $b$ to be predicted by the following function:

$$b = \left[1 - \exp\left(-0.3 \frac{\left(f_c / f_{thre}\right)}{k}\right)\right] \times \left(r \frac{f_c}{f_{thre}}\right)^r \tag{2}$$

where $r = \chi^{-1}$= particle size ratio, d/D and $k = 1 - r^{0.25}$. Since sand and fines are generally not single-size materials, D/d was generalized to $D_{10}/d_{50}$, where the subscripts denote fraction passing. $f_{\text{thre}}$ can be obtained from the experimental data, where available, as outlined in Rahman et al. (2009). However, as an initial approximation, $f_{\text{thre}}$ can be taken as 0.30, but it may be determined more reliably using the following equation developed by Rahman *et al.* (2009).

$$f_{thre} = 0.40\left(\frac{1}{1+e^{\alpha - \beta\chi}} + \frac{1}{\chi}\right) \tag{3}$$

The parameters $\alpha$ and $\beta$ are determined by curve fitting to eight databases for $\chi$ in the range of 2 to 42, and this gave $\alpha$ = 0.50 and $\beta$ = 0.13. With this approach, Rahman and co-workers demonstrated that for any $f_c < f_{\text{thre}}$, the SS data points can be described by a single trend curve in the $e^*$-log($p'$) space. This single trend curve will be referred to as equivalent granular steady state line, EG-SSL. If one can achieve a single EG-SSL in the $e^*$-log($p'$), then the definition of $\psi$, can be generalized to $\psi^*$, by replacing $e$ with $e^*$ and SSL with EG-SSL as shown in Fig. 1b (Rahman and Lo, 2007). Studies on Sydney sand with fines showed that, for $f_c < f_{\text{thre}}$, $\psi^*$ could be used to predict the types of undrained behaviour as shown in Fig. 1 (Rahman and Lo, 2007, Rahman, *et al.*, 2011) and instability stress ratio, $\eta_{\text{IS}}$ (Mizanur and Lo, 2012).

These two powerful unifying propositions: state dependent constitutive relation and equivalent granular state are recently combined by merely substituting $e^*$ and $\psi^*$ for $e$ and $\psi$ into the equations of the aforementioned model by Li and Dafalias (2000) for clean sands to obtain a constitutive model for sand with fines that can capture the influence of $f_c$, irrespective of density and stress states (Rahman, *et al.*, 2014b). The formulation of the model is discussed below.

## Constitutive Relations

The additive decomposition of strain increment $\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^{e} + \mathrm{d}\varepsilon_{ij}^{p}$ is assumed, where superscripts "*e*" and "*p*" denote elastic and plastic components, respectively. The model constitutive equations are presented in the following sub-sections.

### *Elastic components*

Based on earlier research (Hardin and Richart, 1963), Rahman, *et al.* (2012) analysed six clean host sands with different $f_c$ and reported that, as an approximation, the influence of fines on elastic shear modulus can be captured by replacing $e$ with $e^*$ and can be presented as following:

$$G^*_e = C_g \frac{(e_r - e^*)^2}{1+e^*}\left(\frac{p'}{p_a}\right)^{n_g} p_a \tag{4}$$

where $C_g$ and $n_g$ are non-dimensional soil parameters, $e_r$ = 2.97 for angular sands and 2.17 for rounded sands, $p_a$ is reference stress of 100 kPa in consistent unit. Assuming a constant Poisson ratio ν, the incremental elastic bulk modulus $K^*_e$ for sands with fines is given by:

$$K^*_e = \frac{2(1+\upsilon)}{3(1-2\upsilon)} G^*_e = \frac{2(1+\upsilon)}{3(1-2\upsilon)} C_g \frac{(e_r - e^*)^2}{1+e^*}\left(\frac{p'}{p_a}\right)^{n_g} p_a \tag{5}$$

Note that superscript '*' in equations (4) and (5) denotes physical quantities computed based on $e^*$ and/or $\psi^*$; similar use of superscript '*' is made throughout this article.

### *Plastic components*

The aforementioned model by Li and Dafalias (2000), a modification of Manzari and Dafalias (1997), is considered in triaxial space. The flow rule i.e. dilatancy equation for the clean sand in Li and Dafalias (2000, 2012) is modified by replacing ψ with ψ*:

$$d^* = \frac{d_0}{M}\left(M \exp(m\psi^*) - \eta\right) \tag{6}$$

where $d_0$ and $m$ are model parameters, independent of $e^*$ and $\psi^*$. At phase transformation, $d^* = 0$ and therefore, $\eta_{PT} = M \exp(m\psi^*)$; $\eta_{PT}$ becomes identical to the failure stress ratio at SS, $M$, at $\psi^* = 0$. The yield function is assumed to be a family of constant stress ratio lines, hence:

$$f(q, p) = q - \eta\, p' = 0 \tag{7}$$

The incremental stress ratio-plastic deviatoric strain relation can be written as:

$$\frac{\mathrm{d}\eta}{\mathrm{d}\varepsilon_q^p} = \frac{K_p}{p'} \tag{8}$$

where $K_p$ is the plastic hardening modulus given by:

$$K^*_p = \frac{h^* G^*_e \exp(n\psi^*)}{\eta}\left(M\exp(-n\psi^*)-\eta\right) \tag{9a}$$

$$h^* = h_1 - h_2 e^* \tag{9b}$$

with $n$, $h_1$, $h_2$ model parameters assumed to be independent of density. Eq.(9a) has the salient feature of any bounding surface (BS) plasticity model, namely the quantity in parentheses represents the "distance" of the current stress ratio $\eta$ from $M^b = M\exp(-n\psi^*)$ that plays the role of the corresponding BS in stress ratio space, and which becomes identical to the critical state stress ratio, $M$ when $\psi^* = 0$. Notice that for $K_p = 0$, it follows from Eq.(9a) that the corresponding $\eta = M^b = \eta_f$ which is the peak stress ratio at that moment when $\psi^*$ is different than zero, and becomes the critical state stress ratio $M$ when $\psi^*=0$. Eq.(9b) is an empirical relationship (Li and Dafalias, 2000) that $h$ dependent on density. The inclusion of $G^*_e$ in Eq.(9a) ensures that the parameter $h$ is non-dimensional by normalizing its magnitude relative to its elastic stiffness.

***Incremental equations***

Recalling that $\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^e + \mathrm{d}\varepsilon_{ij}^p$, one can express total deviatoric strain by:

$$\mathrm{d}\varepsilon_q = \left(\frac{1}{3G^*_e} + \frac{1}{K^*_p}\right)\mathrm{d}q - \frac{\eta}{K^*_p}\mathrm{d}p' \tag{10}$$

The total volumetric strain increment is now given by:

$$\mathrm{d}\varepsilon_v = \frac{\mathrm{d}p'}{K^*_e} + d^*\mathrm{d}\varepsilon_\mathrm{q} = \frac{\mathrm{d}p'}{K^*_e} + d^*\frac{\mathrm{d}q - \eta\mathrm{d}p'}{K^*_p} = \frac{d^*}{K^*_p}\mathrm{d}q + \left(\frac{1}{K^*_e} - \frac{d^*\eta}{K^*_p}\right)\mathrm{d}p' \tag{11}$$

where $G^*_e$, $K^*_e$, $d^*$, $K^*_p$ are function of $e^*$ and $\psi^*$.

## Model Specific Parameters

Four SS model parameters are much discussed in Rahman and Lo *et. al* (Mizanur and Lo, 2012, Rahman, *et al.*, 2014a, Rahman and Lo, 2014). Seven other model specific parameters were needed. In order to provide a stringent evaluation of the predictive capability of the proposed model, these model parameters were determined from drained tests on specimens of a single $f_c$, and the undrained behaviour for different initial states and $f_c$ were then predicted and compared with test results. Furthermore, it is also desirable that most of these drained tests were conducted independent of the model development. As a result of such considerations, the only suitable independent data source for Sydney sand with fines was (Bobei, *et al.*, 2009) which covered only 10% fines. Four drained tests were extracted from this source to determine model parameters as presented in Table 1.

Table 1: Model parameters obtained from drained tests.

| Elastic | | Steady state | | Dilatancy | | Hardening | | BS | |
|---|---|---|---|---|---|---|---|---|---|
| $C_g$ | 186 | $M$ | 1.305 | $d_0$ | 1.06 | $h_1$ | 1.30 | $n$ | 0.95 |
| $\nu$ | 0.30 | $e_{\text{lim}}$ | 0.920 | $m$ | 0.50 | $h_2$ | 0.60 | | |
| | | $\Lambda$ | 0.0375 | | | | | | |
| | | $\xi$ | 0.60 | | | | | | |

## Model Prediction

Model predictions of S-MII-15-03 and S-MII-20-04 for $f_c$ of 15% and 20%, $p'_0$ of 600kPa and 1100kPa, $e_0$ of 0.655 and 0.560, $e^*_0$ of 0.896 and 0.858, and $\psi^*_0$ of 0.086 and 0.096 respectively were compared with test results in Fig. 2a&b; where subscript ‘0’ represent a state before undrained shearing. Both tests showed flow behaviour and predictions were very close to experimental data. The predicted $q$-$\varepsilon_q$ responses, as shown in Fig. 2b, are also in good agreement with the experimental responses. Furthermore, the overall quality of the predictions was independent of $f_c$. However, despite the manifestation of the same flow behaviour, their global void ratio, $e_0$ reduced with $f_c$ and attained a low value of 0.560 for the test with $f_c$=20%. This indirectly suggests that, if the response is interpreted based on $e_0$, an increase in $f_c$ leads to a reduction in liquefaction resistance and the model is able to capture this feature for sand with fines behaviour.

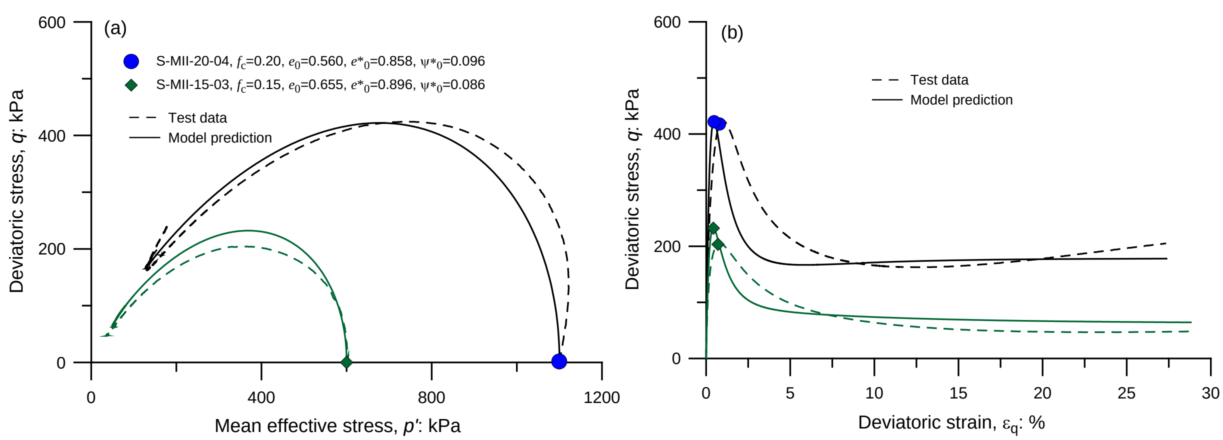


Figure 2: Prediction of liquefaction behaviour for different $e_0$ and $f_c$; (a) effective stress path, (b) $q$-$\varepsilon_q$ responses.

The model prediction for LF behaviour for S-MII-15-05 with 15% $f_c$ is presented in Fig. 3a. Evidently, the prediction for ESP was good. The evolution of states in terms of $\psi^*$ for both experiment and model prediction is shown in $\psi^*$-$\varepsilon_q$ space in Fig. 3b. The $\psi^*$ at the start was +0.056 and then it crossed the EG-SSL to a negative value at PT and then came back to EG-SSL at SS. Evidently, model captured this feature. In an attempt to capture the type of undrained behaviour with respect to initial $\psi_0$ or $\psi^*_0$, Rahman and Lo (2014) identified an overlapping range of $\psi_0$ or $\psi^*_0$ that can exhibit either F or LF behaviour. To evaluate this observation, a synthetic simulation for $f_c$=15%, $e^*_0$=0.903 and $p'_0$=300kPa is presented in Fig. 3a&b. Although, this synthetic simulation has $\psi^*_0$=0.056, exact same as S-MII-15-05, exhibited F behaviour. The $\psi^*$ of simulation, slowly, approached SS to merge with the EG-SSL without crossing it (Fig. 3b). Therefore, the model can capture the experimental observation of either F or LF for same $\psi^*_0$.

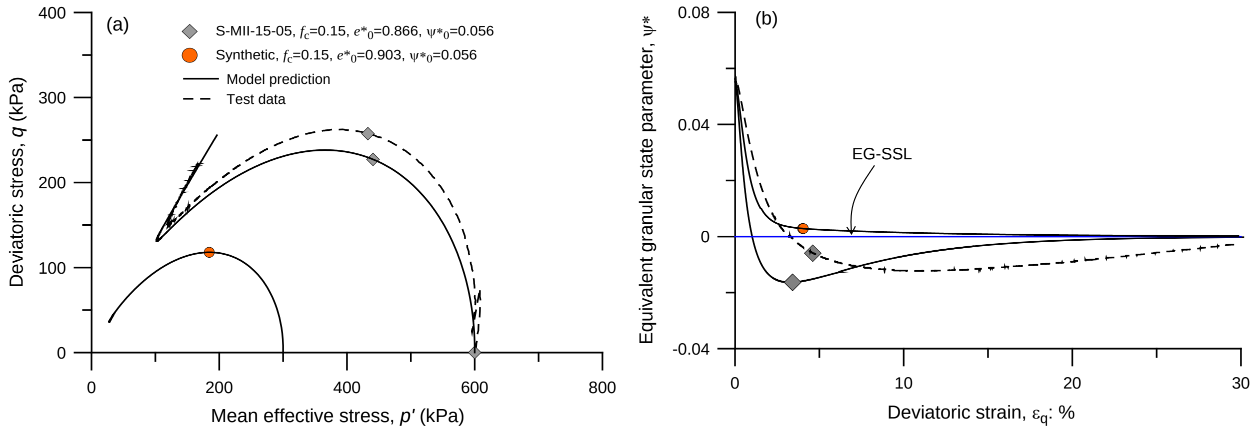


Figure 3: (a) effective stress path for LF and F, (b) $\psi^*$-$\varepsilon_q$ responses.

## Conclusions

Two powerful unifying propositions: state dependency in constitutive relations and equivalent granular state are combined by merely substituting $e^*$ and $\psi^*$ for $e$ and $\psi$, respectively, into the equations of a constitutive model by Li and Dafalias (2000) for clean sands to obtain a constitutive model for sand with fines that can capture the influence of $f_c$, irrespective of density and stress states (Rahman, *et al.*, 2014b). The model was then evaluated with counter-intuitive but significant feature of undrained behaviour for sand with fines. The model was able to simulate:

- A well established experimental observation that $f_c$ as much as $e$ controls sand with fines behavior,
- Recent experimental observation that shows that the same initial states in terms of $\psi_0$ or $\psi^*_0$ may exhibits either flow or limited flow behavior,
- Limited flow behaviour implied by an inherent state evolution in terms of $\psi$ or $\psi^*$ that crosses SSL or EG-SSL, intermediately, to eventually reach a negative $\psi$ or $\psi^*$ and then turn back to SS at $\psi$ or $\psi^*$=0.

At present this model does not consider fabric evolution and thus has certain limitations.


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