6th International Conference on Earthquake Geotechnical Engineering
1-4 November 2015
Christchurch, New Zealand

A Nodal Discontinuous Galerkin Method for Non-linear Soil Dynamics

E. D. Mercerat[1], N. Glinsky[2]

ABSTRACT

We investigate the potential capabilities of the discontinuous Galerkin method (DG-FEM) for non-linear site response analysis. The method combines the geometrical flexibility of the finite element method, and the high parallelization potentiality and the capabilities for accurate simulations of strongly non-linear wave phenomena of the finite volume technique. It has been successfully applied to elastic, visco-elastic and anisotropic media. The natural next step is to extend the method to non-linear soil rheologies.

We develop a discontinuous Galerkin method (nodal approach) for seismic waves in heterogeneous non-linear 1D media. The method is based on high-order Lagrangian interpolation within elements, upwind fluxes, and a fourth-order Runge-Kutta time scheme. The parallel Iwan model is used to account for the non-linear soil behavior with hysteresis loops based on extended Masing rules. Comparison with different numerical methods shows satisfactory results for some canonical cases, at least for strains lower than 1%. Validation with real kik-Net data is work in progress within the Prenolin benchmark project (see this volume).

Introduction

In the recent years, advances in computer architectures render large-scale seismic wave propagation simulations feasible in heterogeneous media. Several numerical methods are available and the final choice is clearly problem dependent, as explained in a recent review by Moczo *et al.* (2014). Among them, the methodologies based in the variational formulation of the dynamical system allow for accurate implementation of boundary conditions through structured or unstructured meshes. This is extremely important for accurate simulation of surface waves, which cause the most earthquake related damage due to their relatively high amplitude and long shaking duration.

Numerical modeling of non-linear site response has been carried out mainly using classical finite differences (Joyner and Chen, 1975; Kramer, 1996; Gélis and Bonilla, 2012) or finite elements techniques (Taborda *et al.*, 2012, Santisi d'Avila *et al.*, 2013). Less known in earthquake geotechnical engineering is the discontinuous Galerkin finite element technique (DG-FEM), which is basically a classical finite element method but without imposing the solution field continuity across neighboring elements of the mesh that produces very accurate and flexible

[1]PhD E. Diego Mercerat, Seismic Risk Team, Direction Territoriale Méditerranée CEREMA, Nice, France. diego.mercerat@cerema.fr
[2]PhD Nathalie Glinsky, IFSTTAR Marne-la-vallée, France. CEREMA, Nice, France. INRIA Sophia Antipolis Mediterrannée France

solvers. On the contrary, this technique needs the calculation of numerical fluxes at the interfaces between neighboring elements which is the key point of the method and is problem dependent. Many different strategies have been proposed in the literature and therefore a huge list of specific techniques is already known. The reader is referred to Cockburn *et al.* (2000) or Hesthaven and Warburton (2008) for detailed information about DG method and its applications in engineering.

In this study we propose a high-order discontinuous Galerkin finite element technique to assess the site response analysis. The technique has been successfully applied to simulate wave propagation in elastic (Dumbser *et al.* 2006, Delcourte and Glinsky, 2015, Etienne *et al.*, 2010, Mercerat and Glinsky, 2015), visco-elastic (Käser *et al.*, 2007, Peyrusse *et al.*, 2014) and anisotropic media (de la Puente *et al.*, 2008). The methodology is well-suited for solving dynamic rupture problems, or any other problem involving discontinuous solutions, in the velocity-stress formulation as the variables are naturally discontinuous at the interface between neighboring elements (de la Puente *et al*, 2009). The difficulty lies in the development of fluxes with good numerical properties. In the case of non-linear soil dynamics, jumps in stress and strain fields can be found at the interfaces with high impedance contrast (Gélis and Bonilla, 2012; Santisi *et al*, 2013). The next step is to extend the discontinuous Galerkin method to non-linear soil rheologies and this is the purpose of this work. We start by the simplest case of 1D site response modeling but the methodology can be easily extended to higher dimensions and is the subject of current research.

Soil column DG-FEM discretization

We solve the elastodynamics system of partial differential equations by choosing as primal variables the velocity and the strain fields. If the medium is 1D and we consider only one transverse degree of freedom, both are scalar fields. The first-order system of equations becomes

$$\begin{cases} \rho\, \partial_t v = \partial_z \sigma \\ \partial_t \varepsilon = \partial_z v \end{cases} \tag{1}$$

where v and ε are respectively the velocity and strain fields, ρ is the bulk density, σ is the stress which is related to the strain by the relationship $\sigma = G\,\varepsilon$, where G represents the shear modulus. The notations ∂_t and ∂_z refer to time and spatial derivatives respectively. Contrary to the linear elastic case, the system becomes non-linear because of the complex dependence of the shear modulus G with strain, i.e. $G = G(\varepsilon, \dot{\varepsilon})$; this particular point will be detailed later.

We consider a soil column discretized by a series of N_e 1D finite elements, named h_i, $i = 1,..,N_e$, as shown in Figure 1a). The spatial interpolation functions are chosen as Lagrange polynomials of order N based at the N+1 interpolation nodes (in this case, the Gauss-Lobatto-Legendre points) within the elements. In the standard element [-1, 1], the Lagrange polynomials of order 3 are shown in Figure 1b). Consequently, the solution fields (for instance v) are expanded in each finite element as

$$v_{|h_i}(z,t) = \sum_{j=1}^{N+1} v_j^{h_i}(t)\, \phi_j^{h_i}(z)\,. \tag{2}$$

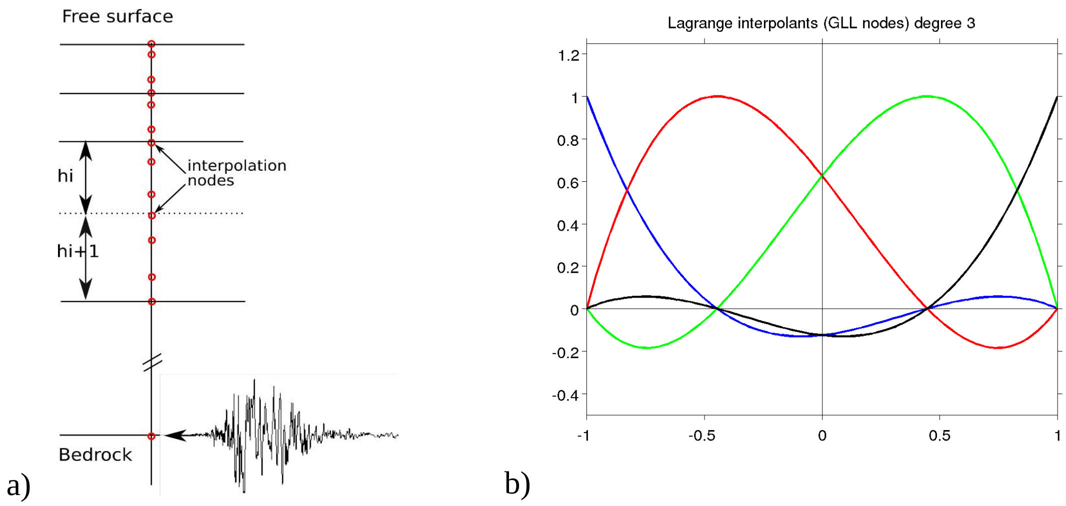

Figure 1: a) 1D soil column discretization with finite elements of degree 3 (4 nodes per element). Free surface boundary condition on top, rigid (or elastic) condition at the bottom. b) Lagrange interpolation polynomials of degree 3 in the standard element [-1,1]

Note that the interpolation functions $\phi_j^{h_i}$ are defined locally in h_i. The system of partial differential equations (1) is multiplied by a test function ϕ and integrated over the domain. Since the test functions are local, after integration by parts, the integrals corresponding to both equations can be written as

$$\int_{h_i} \rho\phi\,\partial_t v\, dz = -\int_{h_i} \partial_z\phi\,\sigma\, dz + \left[\phi\,\sigma^*\right]_{z^-}^{z^+}, \qquad \int_{h_i} \phi\,\partial_t\varepsilon\, dz = -\int_{h_i} \partial_z\phi\, v\, dz + \left[\phi\, v^*\right]_{z^-}^{z^+}, \tag{3}$$

where the brackets correspond to integrals on the boundary of the element (reduced to two nodes in 1D), that is the difference of the fluxes at both boundaries z^+ and z^- of the element. The calculation of these terms is not straightforward since the approximation is discontinuous, then v and σ are not defined at the interface but on both sides. Following a technique proposed by Hesthaven and Warburton (2008) for the Maxwell equations, we propose to approximate the fluxes at the interface between elements by a standard upwind scheme:

$$\sigma^* = \frac{1}{\left\{\frac{1}{\sqrt{\rho G}}\right\}}\left\{\frac{\sigma}{\sqrt{\rho G}}\right\} + \frac{1}{2}[[v]] \qquad \text{and} \qquad v^* = \frac{1}{\left\{\sqrt{\frac{\rho}{G}}\right\}}\left\{\sqrt{\frac{\rho}{G}}\,v\right\} + \frac{1}{2}[[\varepsilon]]$$

(4)

where the mean values {.} and jumps [[.]] at the interfaces between elements are defined by

$\{a\}=\frac{a^{+}+a^{-}}{2}$ and $[[a]]=a^{-}-a^{+}$. We can impose different boundary conditions by defining exterior ghost states (σ^{+},v^{+}). At the free surface (Neumann boundary condition), we impose $(\sigma^{+},v^{+})=(-\sigma^{-},v^{-})$. At the bottom of the model, we may find elastic, rigid or absorbing boundary conditions (inhomogeneous Dirichlet conditions). Therefore we should impose $(\sigma^{+},v^{+})=(I\sigma^{-},-Iv^{-}+Ts)$, where I is the impedance contrast between the soil layer and the bedrock, T is the transmission coefficient and $s=s(t)$ is the input motion (e.g. weakly imposed velocity).

After introducing the approximations of Equation (2) in the integrals of Equation (3) and transforming each element to the standard element [-1, 1], we obtain two types of integrals that are calculated analytically. The time approximation is carried out by a fully explicit fourth-order Runge-Kutta method.

Non-linear soil model

The non-linear Iwan model (Iwan, 1967) is implemented by means of a series of Iwan parallel elements composed of a spring and a perfect plastic slider. Then the non-linear model parameters are H spring stiffnesses and H_r threshold stresses per interpolation node. Even if the model allows to fit any modulus reduction curve (e.g. by spline interpolation), we decide here to fit an hyperbola to the shear modulus degradation curve by fixing the lower and upper strain limits and the reference strain (i.e. strain for which the modulus reduction is 50%). The model parameters are set at the beginning of each simulation. Up to now there is no constraint on the hysteretic damping of the Iwan model, then it is expected the calculated seismograms are overdamped, especially when high strains are mobilized.

As the strain is a primal variable of the scheme, it is straightforward to firstly correct the stresses corresponding to the level of strain obtained at the previous time step, calculate the numerical fluxes and advance in time the system of Equations (3). The algorithm reads,

```
for each timestep
 for each element
  ε(t)<- ε(t-1), v(t)<- v(t-1)
  if NL element
   σ=Iwan(ε)
  else
   σ=Gε
  endif
  call Runge-Kutta(dt,v,σ,ε,G(ε))
  if NL element then G=G(ε)
 next element
next timestep
```

The *Iwan* routine calculates the stress from the strain taking into account unloading effects (i.e. hysteresis loops) by means of generalized Masing rules. The *Runge-Kutta* routine is in charge of the numerical flux calculation (right-hand side of equations (3)) and then it advances in time the scheme. It remains unchanged for linear or non-linear elements what ease the implementation in existing DG codes. The proposed scheme shows to be robust up to strains levels around 1%. For larger strains, simulations carried out with real accelerograms as input motions have shown some instabilities (spiky behavior) in the output time histories when the shear modulus reduction goes beyond 90%. As already mentioned, the development of numerical fluxes is a difficult task, especially for non-linear systems of partial differential equations. If the standard upwind scheme of equation (4) has been successfully verified against other numerical methods (see Regnier *et al* 2015) for low to moderate strain values, it is necessary to improve it for larger strains. Current research is focused on this subject.

Numerical results

Verification against SEM

As a first step, the verification of our 1D non-linear DG solver is accomplished by comparing the results with the ones calculated using a solver based on the Spectral Element Method (SEM) that also implements the Iwan non-linear model for soils (Oral E, 2015 pers comm). We have run a simple simulation of a non-linear layer over a rigid half-space with a Ricker wavelet of 4 Hz central frequency as input motion at the bottom of the soil layer (weakly imposed velocity). The soil characteristics are shown in Table 1.

In Figure 2, we present the surface seismograms for both simulations indicating a quite good match all along the simulation, though some differences can be appreciated at the later times probably due to the different time schemes used (explicit 2nd order Newmark for SEM, 4th order Runge-Kutta for DG-FEM).

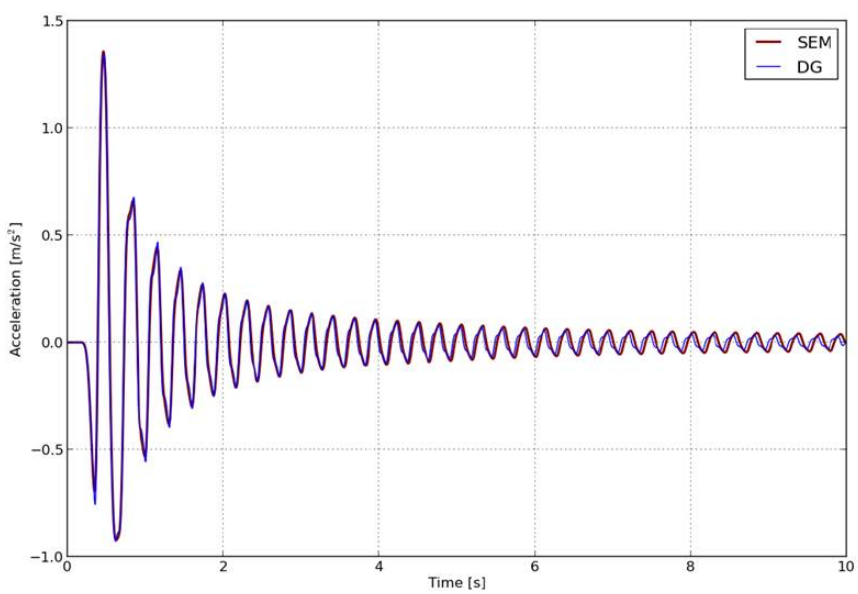

Figure 2: Comparison of acceleration time histories at the surface of the model for the spectral element method (SEM) and the discontinuous Galerkin (DG) scheme

Further verification tests with other well documented non-linear numerical simulation codes can be found in the present volume (Regnier et al, 2015). We use the present version of the solver for the validation phase of the Prenolin project with real data from the kik-Net network (Regnier et al, 2015).

Effect of different G-γ decay curves

We turn now to a comparison of the effect of different non-linear soil properties in the waveform characteristics at the free surface. The soil column is composed of a soil layer overlying an elastic half space (bedrock) with mechanical properties shown in Table 1. The linear elastic properties of the soils are kept constant for the simulations. On the contrary, three different modulus reduction curves will be used as shown in Figure 3. The soil column of 20 m depth is discretized by 10 finite elements with interpolation degree 4. The time step is kept fixed at 1 10^{-4} seconds. As input motion, we use a Ricker wavelet of 4 Hz central frequency to better visualize the effect of the non-linear soil response.

Table 1. Mechanical characteristics of the soil column

	Z [m]	Density [kg/m3]	Vs [m/s]	
Soil	20	2000	300	Non-linear (soil 1, 2 and 3)
Bedrock	inf	2500	1000	Linear

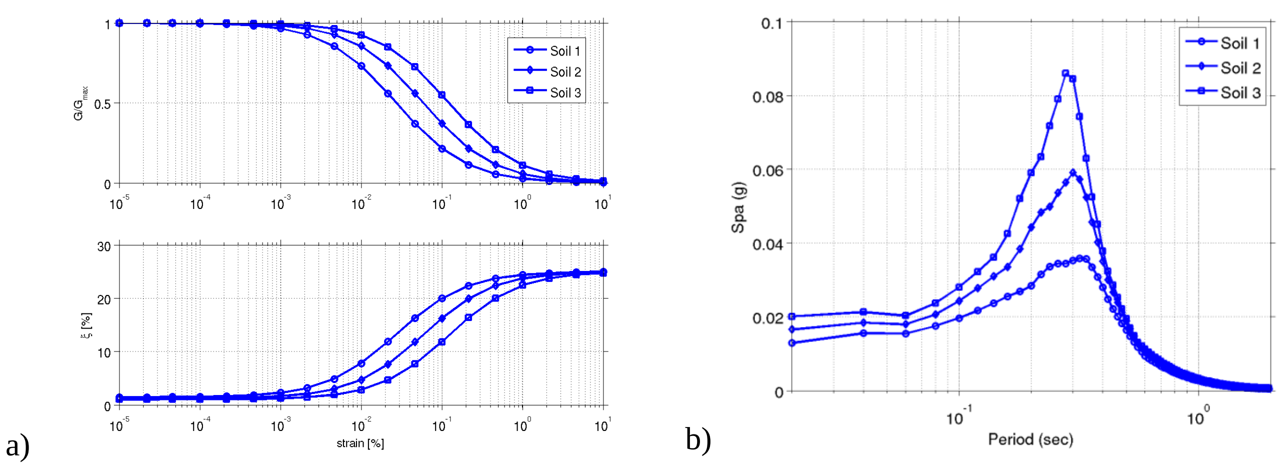

Figure 3 : a) Non-linear modulus decay and damping curves for three different soils. b) Response spectra for the three different soils at the surface of the model. The spectral peak is shifted toward longer periods as the soil gets weaker for a given strain level (Soil 3 to Soil 1).

The meshes are constructed supposing a minimum propagated wavelength of 1/10 of the minimum elastic wavelength , where is the initial shear wave velocity and the maximum frequency of the input motion (10 Hz in this case).

In the acceleration time histories of Figure 4, we can clearly observe the distortion of the input wavelet (shown in gray at each panel) while propagating within the non-linear soil layer. The hysteretic damping effect is also put forward as there is no other damping (numerical or visco-elastic) implemented in the scheme. For these simulations rigid boundary conditions are used. The response spectra of these three simulations are shown in Figure 3b

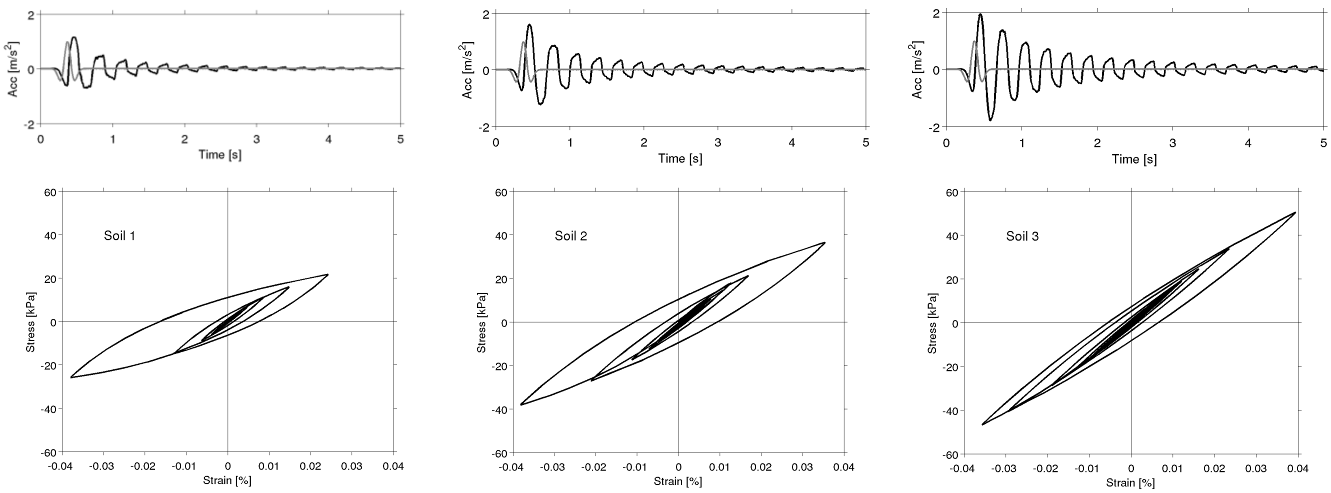

Figure 4 : Surface accelerations (top row) and stress-strain hysteresis loops (bottom row) for the three different soils of Figure 3.

Conclusions

We present a nodal discontinuous Galerkin method for seismic waves in heterogeneous non-linear 1D media. The method based on standard upwind fluxes and classical 4^{th} order Runge-Kutta time scheme seems to be robust, at least for strain levels lower than 1%. We verified our approach by comparing the simulated time histories with the ones obtained using a SEM solver for the same soil column and non-linear soil model. The methodology possesses in principle many advantages with respect to more classical methods (finite differences, standard finite elements) for solving non-linear partial differential equations in realistic 3D media in the presence of non-regular solutions. The extension to multi-component wave propagation and effective stress analysis is subject of current research.

Acknowledgments

The authors greatly acknowledge the simulations carried out by Elif Oral that helped to validate the implementation of the non-linear Iwan model in our DG solver.

References

Cockburn B., Karnadiakis G., Shu C., Discontinuous Galerkin Methods, Lecture Notes in Computational Science and Engineering, Springer, 2000.

De la Puente J., Käser M., Dumbser M., Igel H., An arbitrary high-order discontinuous Galerkin method for elastic waves on unstructured meshes- IV. Anisotropy, Geophys. J. Int., **169**(3), 1210-1228, 2007.

De la Puente J., Ampuero J.P. and Käser M., Dynamic rupture modeling on unstructured meshes using a discontinuous Galerkin method, *J. Geophys. Res*., **114**, B10302, 2009.

Delcourte S., Glinsky N., Analysis of a high-order space and time discontinuous Galerkin method for elastodynamic equations. Applications to 3D wave propagation, *ESAIM Math. Mod. Numer. Anal.,* 49, 1085-1126, 2015.

Dumbser M., Käser M., An arbitrary high-order discontinuous Galerkin method for elastic waves on unstructured meshes- II. The three-dimensional isotropic case, *Geophys. J. Int*., **167**(1), 319-336, 2006.

Etienne V., Chaljub E., Virieux J., Glinsky N., An hp-adaptive discontinuous Galerkin finite element method for 3D elastic wave modelling, *Geophys. J. Int.,* **183**(2), 941-962.

Gélis C., Bonilla L.F., 2-D P-SV numerical study of soil-source interaction in a non-linear basin, *Geophys. J. Int*., **191**, 1374-1390, 2012.

Hesthaven J., Warburton T., *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis and Applications*, Springer, 2008.

Iwan W.D., On a class of models for the yielding behavior of continuous and composite systems, *J. of Appl. Mech*., **34**(3), 612-617, 1967.

Joyner W.B. and Chen, A.T.F. Calculation of nonlinear ground response in earthquakes. *Bull. Seism. Soc. Am.*, **65**, 1315-1336, 1975

Käser M., Dumbser M., de la Puente J., Igel H., An arbitrary high-order discontinuous Galerkin method for elastic waves on unstructured meshes - III. Viscoelastic attenuation, Geophys. J. Int., **168**, 224-242, 2007.

Kramer, S. *Geotechnical Earthquake Engineering,* Prentice Hall International, 1996.

Mercerat E.D., Glinsky N., A nodal high-order discontinuous Galerkin method for elastic wave propagation in arbitrary heterogeneous media, *Geophys. J. Int,* in press, 2015.

Moczo P., Kristek J., Galis M., *The Finite-Difference Modelling of Earthquake Motions: Waves and Ruptures*, Cambridge Univ. Press, 2014.

Oral E., Bonilla F., Gelis C and Delavaud E., The Spectral Element Method for non-linear soil rheologies. Pers. Communication, 2015.

Peyrusse F., Glinsky N., Gélis C., Lanteri S., A nodal discontinuous Galerkin method for site effects assessment in viscoelastic media – verification and validation in the Nice basin, *Geophys. J. Int*., **199**, 315-334, 2014.

Taborda R., Bielak J., Restrepo D., Earthquake Ground Motion Simulation including Nonlinear Soil Effects under Idealized Conditions with Application to Two Case Studies, *Seismol. Res. Lett*., **83**, 1047-1060, 2012.

Regnier J., Bonilla F., Bard P.Y., Bertrand E., Hollender F., Kawase H., Sicilia D. and the Prenolin participants. PRENOLIN project : a benchmark on numerical simulation of 1-D nonlinear site effects. *6ICEGE*, Christchurch, New Zealand. 2015

Santisi d'Avila M.P., Semblat J.-F., Lenti L., Strong ground motion in the 2011 Tohoku Earthquake : A one directional three-components modeling, *Bull. Seism. Soc. Am*., **103**(2B), 1394-1410, 2013.